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# VERTICAL CONTROL NETWORK ADJUSTMENT USING DIFFERENT LEAST SQUARE METHODS

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## Abstract

In all measurable quantities, true values are not known. Therefore, observations are assumed to contain some sort of error. Thus, redundant observations are carried out to obtain the best-estimated value by applying a suitable mathematical model. The least square method is well-known and widely applied in surveying to adjust control networks.

The term levelling in surveying is referred to the process of height difference determination. Thus, the elevation of points can be calculated relative to a reference datum. The datum of survey control networks sometimes cannot be defined because there is no reference or control point available. In this case of a control-free network, the solution can be obtained by what is known as the general solution of free datum networks. Usually, the datum in free control networks is defined by additional constraints (datum constraints) to solve the problem of a singularity of the normal equation matrix.

In this research work, five different least square adjustment methods are applied to a levelling network involving the classical observation equations method, datum constraint equations method, inversion of the full normal equation, Helmert-Wolf decomposition method, and observational definition of datum method. The results obtained for the unknowns were similar in all methods used. This implies that the datum parameters can be directly fixed using constraint equations. It can also be used to fix any of the unknowns when required. Observations defining the datum positively reflected on the results since it increases the redundant observations.


**Keywords:** Constrain Equation, Control Network, Datum, Global Positioning, System (GPS), Helmart-Wolf Decomposition Method, Inversion of Full Normal Equations, Least Square, Levelling, Observation Equation.

## Introduction

In surveying, a control network is the framework of survey stations whose coordinates have been precisely determined. It can be either horizontal control or vertical control. Horizontal control can be carried out using traverse, triangulation, trilateration intersection, or resection. Vertical control can be done using differential levelling, trigonometric levelling, or the like. Recently the Global Positioning System (GPS) has gradually been replacing these conventional methods in establishing both the horizontal and vertical control networks.

In practical survey networks, it is usual to observe more than the strict minimum number of observations required to solve for the unknowns. The extra observations are 'redundant' and can be used to provide an independent check, but all the observations can be incorporated into the solution of the network if the solution is done by the least squares. All observations have some sort of random error so any practical set of observations will not perfectly fit any chosen set of variables for the unknown points. If all the observations are to be used, then they will have to be adjusted so that they fit with the calculated network.

Accuracy of measurement refers to the closeness of the mean of measurement results to the true value and the degree of agreement within individual measurement results. On the other hand, precision is the level of closeness of agreement of a set of measurement results of the same observable among themselves.

The datum of survey control networks sometimes cannot be defined because there is no position of fixed points provided. In these types of networks, the solution can be obtained by what is known as the general solution of free datum networks. The datum in free networks is defined by additional datum constraints to solve the problem of the singularity of the normal equation matrix. To solve the problem of datum defect, the unknown parameters of the datum are required to be fixed. These constraints can be established for a part or all points of the network to define the reference frame of the control network in

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several ways. The adjustment obtained using full datum constraints is called free or relative datum adjustment.

The free network (inner constraint) adjustment model provides a means of solving rank-deficient systems through the imposition of a particular set of minimal constraints that do not limit the freedom of the network to translate, rotate, or change size. This method of adjustment has been widely used and is reputed to remove the problem of datum definition. Without the tool of free network adjustment, movements of the datum points could not be detected or might lead to erroneous conclusions. This adjustment method imposes the following constraints on the adjustment.

Traditionally the observation equations method is adopted for the adjustment process. The datum problem is solved by removing the respective columns from the coefficient matrix. In this research work the datum problem is handled by additional constraint equations. Furthermore, constraint equation(s) defining the required function of the unknowns are investigated.

## Mathematical Approach

In surveying network computations, linearized equations involving observations are usually reduced, to take the form of:

$$Ax = b + v$$

where,
*x* is a vector of the variables to be computed,
*b* is a vector which relates to the observations,
*A* is a matrix of coefficients,
*v* is a vector of the residuals.

If the weight matrix *W* applies the right level of importance to each of the observation equations and describes how well the observations fit the computed variables, the *W* matrix can be written as:

$$W_{n\times n} = \begin{bmatrix} \frac{1}{\sigma_1^2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\sigma_n^2} \end{bmatrix}$$

Where, $\sigma_i$ is the standard error. However, if the observations are of equal precision, then the weight matrix becomes an identity matrix i.e. $W = I$.

The least squares principle states that the best estimate values of the variables (unknowns) can be obtained by minimizing the sum of the squares of the weighted residuals of all the observations, i.e. minimize $v^T W v$ subject to $Ax = b + v$. Then the solution is usually referred to as the normal equations and can be obtained in such a way that:

$$x = (A^T W A)^{-1} A^T W b \rightarrow Eq.(1)$$

To define the quality of measurements used precision and reliability of control networks, the precision can be obtained from the variance-covariance matrix of the estimated parameters (unknowns), which is defined as follows:

$$C_{\bar{X}} = (A^T W A)^{-1} \rightarrow Eq.(2)$$

Where A is known as the design matrix and W is the weight matrix.

The standard errors are obtained by taking the square root of the corresponding diagonal elements of $C_{\bar{X}}$. These quantities are a function of the arbitrarily chosen datum parameters.

Observations must often satisfy established numerical relationships known as geometric constraints. In differential levelling loops, the elevation differences should sum to a given quantity. However, because the geometric constraints rarely meet perfectly, therefore, the data need to be adjusted.

Equations that relate observed quantities including observational residuals to independent unknown parameters are called observation equations and therefore, the total number of equations must be equal to the number of observations.

In the case of using the Constraint Equation method, observation equations with additional parameters can be written as follows:

$$F(\hat{X}, \hat{S}) = \hat{L}$$

Where, $\hat{X}\ and\ \hat{S}$ are the true vectors of parameters and additional parameters respectively.

$\hat{X} = x^0 + x$ and $\hat{S} = s^0 + s$ where, $x^0$, $s^0$ are the approximate values and $\hat{L} = l + v$

To calculate the most probable value of unknowns or parameters $\hat{x}$ , the following equation can be applied.

$$\begin{aligned} x = [(A^T W B)^T - (B^T W B) \\ \times (A^T W B)^{-1} (A^T W A)]^{-1} \\ \times [(B^T W b) \\ - (B^T W B) \\ \times (A^T W B)^{-1} (A^T W b)] \\ \rightarrow Eq.(3) \end{aligned}$$

The additional parameters $\hat{S}$ can be calculated as follows.

$$\hat{S} = (A^T WB)^{-1} \times [(A^T Wb) - (A^T WA) \times \hat{x}] \rightarrow Eq.(4)$$

Where,

$$A_{n\times m} = {\partial F}/{\partial X}\,|_{X^0,S^0}$$
$$B_{n\times k} = {\partial F}/{\partial X}\,|_{X^0,S^0}$$

$n$, is the number of variables, $m$ is the number of observations, $k$ is the number of additional parameters and S is the true vector of additional parameters of dimensions ($k\ x1$).

In the Inversion of the Full Normal equation, the constraint equations can be written not only to fix the datum but also to fix any required function of the estimated quality. In this method, the solution for the least squares estimation of the parameters $\hat{x}$ is given by the below equation.

$$\begin{bmatrix}\hat{x}\\ k\end{bmatrix} = \begin{bmatrix}A^T WA & R\\ R^T & 0\end{bmatrix}^{-1} \begin{bmatrix}A^T Wb\\ c\end{bmatrix} \rightarrow Eq.(5)$$

The Helmert-Wolf method depends on derivation based on Helmert-Wolf decomposition. It could be used only when the inverse of the matrix $N = A^T WA$ can be found. Otherwise, this method cannot be used.

If $N = (A^T WA)$, $u = (A^T Wb)$, D= $(A^T WB)$, $D^T = (B^T WA)$, $M = (B^T WB)$, and $R = (D^T - MD^{-1} - MD^{-1}N)$, then the solution for the least squares estimation of the parameters $\hat{x}$ and their corresponding variance-covariance matrix $C_{\hat{x}}$ is given by the following equations:

$$\hat{x} = N^{-1}[u - R(R^T N^{-1} R)^{-1} R^T N^{-1} u - C] \rightarrow Eq.(6)$$

and the coefficient matrix of residuals $C_{\hat{x}}$

$$C_{\hat{x}} = (A^T WA)^{-1} \rightarrow Eq.(7)$$

To define the datum by Fixing an Observation method, the mathematical model relating the parameters, and any set of observations can be written as:

$$F(\hat{x}) = \bar{l}$$

Where, $\hat{x}\ and\ \bar{l}$ are the true values of the parameters and observations respectively.

Then the solution for the least squares estimation of the parameters $\hat{x}$ is obtained from the general equation $Ax = b + v$ .

## Results and Analysis

In this research work, a levelling network of eleven lines running through seven stations was assumed to be observed with a specific standard error of observations. Then, this network was adjusted using five methods of the least square. The result of each method was investigated and compared with the other methods as shown in the following paragraphs.

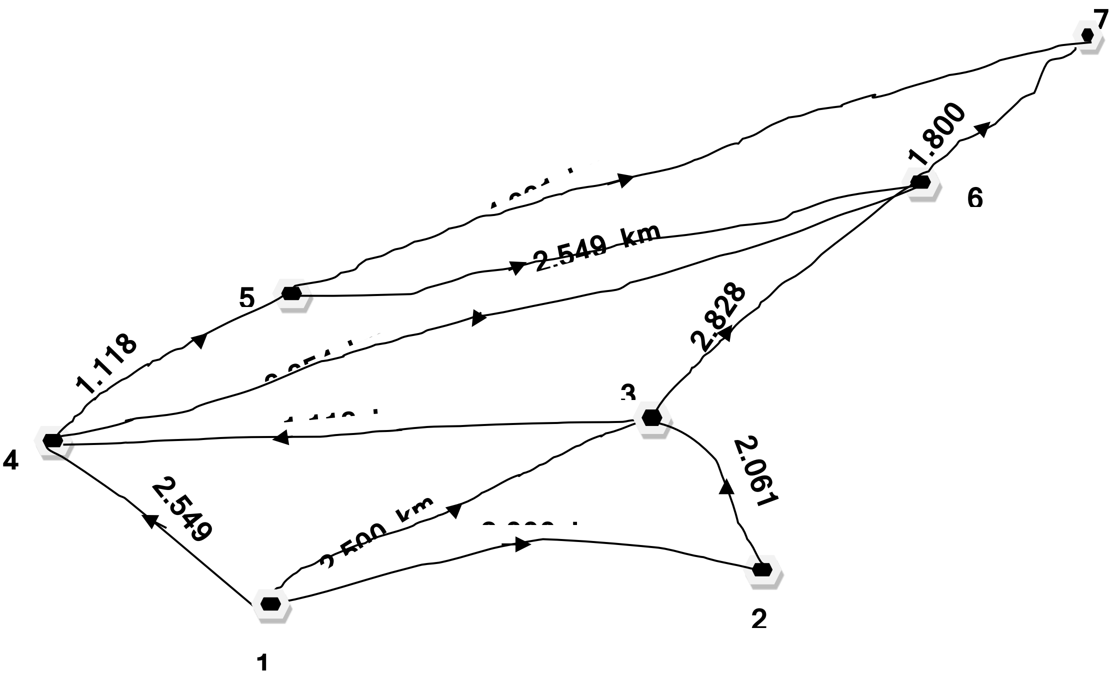
Table (1) hereunder illustrates the observed difference in level between the network points through each line.

**Table 1: The Network Observations and Standard Errors**

| ***Line*** | ***Distance (km)*** | ***Δh (m)*** | ***σ (m)*** |
|---|---|---|---|
| 1-2 | 2.000 | 1.510 | 0.014139 |
| 1-3 | 2.500 | 2.775 | 0.015808 |
| 1-4 | 2.549 | 1.618 | 0.015963 |
| 2-3 | 2.061 | 1.250 | 0.014353 |
| 3-4 | 1.118 | - 1.175 | 0.010569 |
| 3-6 | 2.828 | 1.322 | 0.016814 |
| 4-5 | 1.118 | 1.800 | 0.010569 |
| 4-6 | 3.354 | 2.480 | 0.018311 |
| 5-6 | 2.549 | 0.700 | 0.015963 |
| 5-7 | 4.031 | 1.876 | 0.020075 |
| 6-7 | 1.800 | 1.200 | 0.013413 |

Figure (1) below is a diagram representing the study levelling network lines.

**Figure 1: Levelling Network Lines**




Observations were first arranged in the form of $Ax = b + v$. This results in forming eleven equations as shown below.

$$
\begin{aligned}
x_2 - x_1 &= 1.510 + v_1 \\
x_3 - x_1 &= 2.775 + v_2 \\
x_4 - x_1 &= 1.618 + v_3 \\
x_3 - x_2 &= 1.250 + v_4 \\
x_4 - x_3 &= -1.175 + v_5 \\
x_6 - x_3 &= 1.322 + v_6 \\
x_5 - x_4 &= 1.800 + v_7 \\
x_6 - x_4 &= 2.480 + v_8 \\
x_6 - x_5 &= 0.700 + v_9 \\
x_7 - x_5 &= 1.876 + v_{10} \\
x_7 - x_6 &= 1.200 + v_{11}
\end{aligned}
$$

These equations are then represented in matrix form as shown hereunder.

$$
\begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1
\end{bmatrix}
\times
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}
=
\begin{bmatrix} 1.510 \\ 2.775 \\ 1.618 \\ 1.250 \\ -1.175 \\ 1.332 \\ 1.800 \\ 2.480 \\ 0.700 \\ 1.876 \\ 1.200 \end{bmatrix}
+
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \end{bmatrix}
$$

From the standard error $\sigma_i$ given in table (1) above, the weight matrix can be written as an 11×11 square matrix ($w$).

$$W = \begin{bmatrix} 5002.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4001.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3924.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4854.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8952.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3537.3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8952.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2982.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3924.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2481.4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5558.6 \end{bmatrix}$$

Then, the above data were adjusted using the least squares applying different methods of lest square as discussed in the following paragraphs.

a. The first adjustment applied for the network under consideration is by using the direct least square solution by defining the datum through fixing point 1 *i.e.* assuming point 1 as a benchmark. This will lead to eliminating the first column in matrix A. Then, the unknowns and standard errors can be computed using the least square equations (1) and (2) as discussed before. Table (2) below represents the obtained results.

**Table 2: Results of the Observation Equation Method**

| ***h*(m)** | $\hat{x}\ (m)$ | ***σ (m)*** |
|---|---|---|
| 2 | 1.5176 | 0.0113 |
| 3 | 2.7754 | 0.0103 |
| 4 | 1.6080 | 0.0112 |
| 5 | 3.4074 | 0.0140 |
| 6 | 4.0993 | 0.0142 |
| 7 | 5.2944 | 0.0174 |

b. Another adjustment applied by using the Datum Constraint Equation method. The constraint used here is fixing point 1. Then, the solution is obtained by using the full A matrix in addition to a constraint equation defining the datum. i.e. $R^T X = 0$ *where* $R^T = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$. The parameters ($\hat{x}$) and their corresponding variance-covariance matrix ($C_{\hat{x}}$) are given by equations (3) and (4) respectively as mentioned above. The results obtained as shown in table (3) hereunder.

**Table 3: Results of Datum Constraint Equation Method**

| ***h(m)*** | $\hat{x}\ (m)$ | ***σ (m)*** |
|---|---|---|
| 1 | 0.0000 | 0.0000 |
| 2 | 1.5176 | 0.0113 |
| 3 | 2.7754 | 0.0103 |
| 4 | 1.6080 | 0.0112 |
| 5 | 3.4074 | 0.0140 |
| 6 | 4.0993 | 0.0142 |
| 7 | 5.2944 | 0.0174 |

c. The third method of adjustment is applied by using the General Solution with Additional Constraints. Here, station one was held fixed together with the difference in height between stations two and three. This solution implies the constraint equations $R_x^T = c$ to be as follows:

$$R_x^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.0000 \\ 1.250 \end{bmatrix}$$

Two approaches were applied to get the solution for the network. Those were the Inversion of the Full Normal Equations method and The Helmert-Wolf method.

**i.** In the Inversion of the Full Normal Equations method the solution for the least squares estimation of the parameters ($\hat{x}$) found by using equation (5). Then, the results obtained are arranged as illustrated in table (4).

**Table 4: Results of Inversion of Full Normal Equation Method**

| ***Point*** | $\hat{x}\ (m)$ |
|---|---|
| 1 | 0.0000 |
| 2 | 1.5176 |
| 3 | 2.7754 |
| 4 | 1.6080 |
| 5 | 3.4074 |
| 6 | 4.0993 |
| 7 | 5.2944 |

**ii.** On the other hand, the Helmart-Wolf solution was obtained by applying equations (6) and (7) as described before. The obtained results are listed in table (5) below

**Table 5: Results of the Helmart-Wolf Method**

| ***Point*** | $\hat{x}\ (m)$ | | ***σ (m)*** |
|---|---|---|---|
| 2 | 1.5176 | | 0.0113 |
| 3 | 2.7754 | | 0.0103 |
| 4 | 1.6080 | | 0.0112 |
| 5 | 3.4074 | | 0.0140 |
| 6 | 4.0993 | | 0.0142 |
| 7 | 5.2944 | | 0.0174 |

d. Finally, the network was adjusted by defining the datum. Here, point 1 was assumed to be observed with a 100.000m reduced level and 0.0316m standard error of observation. *i.e.* weight of 1000.0000. Then, the variables can again be calculated by applying equations (1) and (2) as discussed before. Table (6) below represents the obtained results.

**Table 6: Results of the Observational Definition of the Datum**

| *Point* | $\hat{x}\ (m)$ | $\sigma\ (m)$ |
|---|---|---|
| 1 | 100.00 | 0.0316 |
| 2 | 1.5176 | 0.0477 |
| 3 | 2.7754 | 0.0455 |
| 4 | 1.6080 | 0.0474 |
| 5 | 3.4074 | 0.0545 |
| 6 | 4.0993 | 0.0550 |
| 7 | 5.2944 | 0.0700 |

Analyzing the results of the five solutions applied and different assumptions used, it can be noted that the results of the variables ($\hat{x}$) were identical. On the other hand, the standard errors of the variables ***(σ)*** were also identical except for the final method - Observational Definition of the Datum - in which the datum was defined by assuming point 1 to be observed from a reference datum with a new standard error. Thus, positively affects the results of standard errors since it increases the redundant observations.

Also, it can be pointed out that adding constraints makes the computational process so complicated. Results furthermore reflect that both free and constraint equations produced the same results.

## Conclusion

This research work attempted to apply and investigate the results of using different methods of the least square adjustment to a vertical control network. These methods include the classic Observation Equation, Datum Constraint Equations, Inversion of Full Normal Equations, Helmart-Wolf decomposition method, and Observational Definition of the Datum. From the results obtained and analysis carried out, the following conclusions can be drawn.

- All least square methods used provide similar results for the unknowns.
- The datum parameters can be directly fixed using constraint equations. It can also be used to fix any of the unknowns when required.
- Observations defining the datum positively reflected on the results since it increases the redundant observations.

## Conflicts of Interest

The authors declare that there are no significant competing financial, professional, or personal interests that might have influenced the performance or presentation of the work described in this manuscript.

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