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DISCRETE STUDY OF GRAPHS WITH METRIC DIMENSION TWO-A CHARACTERIZATION

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Abstract

There are several methods to associate a code for a vertex. For example, Paul F. Tsuchiya assigned the codes by decomposing the network into sub net-works. The method of approach in random and the code associated depends only on the number of sub divisions. However, a mathematical approach for the assignment of distinct codes given by F. Harary et al and further studied by various other authors, purely depends on the other invariants associated with the network namely, diameter, distance between two vertices etc. The codes generated by the methods given in the above references can easily be implemented to locate any vertex in the graph network.

Keywords: Graphs, Metric Dimension.

Introduction

Every network can be viewed as a graph in which the vertices represent the pro-cessors and an edge between any two vertices indicate the connection between the processors corresponding to the vertices. In Samir Khuller et al, navigations are studied in a graph-structured framework in which the navigating agent (the robot) moves from vertex to vertex of a graph space. The robot can locate itself by the presence of the distinct codes assigned for the vertices of the graph. There are several methods to associate a code for a vertex. For example, Paul F. Tsuchiya assigned the codes by decomposing the network into sub net-works. The method of approach in random and the code associated depends only on the number of sub divisions. However, a mathematical approach for the assignment of distinct codes given by F. Harary et al and further studied by various other authors, purely depends on the other invariants associated with the network namely, diameter, distance between two vertices etc. The codes generated by the methods given in the above references can easily be implemented to locate any vertex in the graph network.

In this chapter, the distance partition of vertex set of a graph G is de ned, with reference to a vertex in it and with the help of the same, characterize the graphs with metric dimension two (i.e. (G) = 2). In the process, develop a polynomial time algorithm that veri es if the metric dimension of a given graph G is two. The same algorithm explores all metric bases of graph G whenever (G) = 2. A bound for cardinality of any distance partite set with reference to a given vertex, whenever (G) = 2 is found.

Throughout this chapter, we write G(V; E) or simply G, to denote a graph on a nite non empty set V of vertices and E of edges. All the graphs considered in this chapter are simple, nite, undirected and connected. For any two vertices u and v, the distance between u and v is denoted by d(u; v), is the length of the shortest path between them. For a given graph G, there are a number of properties related to the distance between two vertices and have been widely studied by various authors.

The present scenario is in the era of communication or the age of pervasive / ubiquitous computing. Identification of any person or system at any given location at any given time based on global positioning systems is expected to be the norm.

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This proliferation of connectivity can be understood by the fact that already the connected systems on the network are capable of reaching nearly twice the population of the earth, even when half the world is not connected. Users expect to be no longer constrained by slow connections, limited speed of computer processors and high costs of storage, for inclusive technology like cloud computing has evolved, thus oaring the unconnected world an inexpensive option to choose the best of features provided by the Information and Communication Technology(ICT).

Researchers are already doing some rethinking on the internet underlying architecture, and are even considering the replacement of the networking equipment and rewriting software to meet the growing demand. One challenge in any reconstruction, though, will be balancing the interests of various constituencies. Recently the national science foundation (NSF) USA has proposed to build an experimental research network known as Global Environment for Network Innovations, or GENI. Several projects are being funded in universities and else where through a program called Future Internet Network Design (FIND).

Science and Engineering

Multi discipline teams and multi discipline areas are words that now a days seem to be important in the scantiest research. Indeed this is even more important if we focus on ancittae scientiae, i.e. sciences, which are used as tools in other sciences such as Mathematics is with respect to Engineering. Mathematicians are usually looking for new varieties of problems and Engineers are usually looking for new arrival solutions. The dialogue of both areas is interesting for both communities and conducive to betterment of science and society.

Nevertheless dialogue is not always easy and requires sucienteorts from both Mathematicians and Engineers. The translation of a problem arising from Engineering to mathematical language requires a deep knowledge of the discipline in which a problem is contextualized. Setting up of assumptions and hypothesis of the model is always a compromise between realism and tractability. Finally the result given by the mathematical models must be validated by using engineering tools for conducting experimental or simulation investigations.

This thesis deals with two such problems of engineering which are useful in communication networks. We model the problem using the theory of graphs, solve it using graph theory concepts and using this concept the design and develop a routing algorithm for a clustered network is presented.

Characterization of Graphs with Metric Dimension Two

The following is the characterization of graphs with metric dimension two.

Theorem. Let G be a graph which is not a path with V (G) = fv1; v2; : : : ; vng and fVi0; Vi1; : : : ; Viki g be the distance partition of V (G) with reference to the vertex vi where ki is the eccentricity of vi, 1 i n. The metric dimension of G is 2 if and only if there exist vertices vi and vj such that jVik \ Vjl j 1 for every k and l with 1 k e(vi) and 1 l e(vj).

Proof. Given vp and vr, jVpq \Vrs j > 1 for some pq and rs implies that there exists at least two vertices, say u1 and u2 in Vpq \Vrs such that d(vp; u1) = d(vp; u2) = q and d(vq; u1) = d(vq; u2) = s and hence u1 and u2 are not resolved by both vp and vr so, jVpq \ Vrs j > 1 for all Pq and rs implies no pair of vertices vp and vr resolves V (G), in other words (G) 6= 2.

Conversely if there exist vp and vr such that jVpq \ Vrs j 1 for all pq and rs, then given any pair of vertices w1 and w2 from V (G) we have w1 2 Vpq1 \ Vrs1 and w2 2 Vpq2 \ Vrs2 where at least pq1 is different from pq2 or rs1 is different from rs2. This implies that w1 and w2 are resolved by at least one of vp and vr. So (G) 2 and in fact, (G) = 2 as G is not a path.

A technology can be sustainably viable only if it can nd widespread use. In order to allow ad hoc networks to achieve commercial success, the scalability problem must be solved. One promising approach is to build hierarchies among the nodes, such that the network topology can be abstracted. This process is commonly referred to as clustering and the substructures that are collapsed in higher levels are called clusters.

The concept of dividing the geographical region to be covered into small zones has been presented implicitly as clustering in the literature. A natural way to map a "standard" cellular architecture into a multi-hop packet radio network is via the concept of a virtual cellular network. Any node can become a cluster head if it has the necessary functionality, such as processing, transmission power and partition the networks based on unique ID. Nodes register with the nearest cluster head and become members of that cluster. A node can communicate with the cluster head if they are within the shortest path with respect to the cluster head. These nodes are called members of that cluster in the network.

Clustering is essentially a multi-leader election problem, which is a modified version of the leader election problem. A leader election algorithm is a process in which a set of distributed nodes (such as ATM, computers, and so on.) tries to identify one node as their leader. In distributed systems, the leader and the other distributed nodes may be called coordinator and participants respectively. The coordinator and its participants make a group in distributed system terminology. The election process is randomized, that is, at every stage of the algorithm those nodes that have survived so far ip a biased coin, and those who received, say a tail, survive for the next round. However, the probability of a particular outcome need not be uniform. The algorithm continues until only a single node remains, which declares itself as a leader. The main problem here is the number of rounds needed to select a leader, or in other words, the time complexity may be high if optimality is a desired criterion.

In ad hoc networks, nodes are distributed randomly and they are identified by their unique IDs. In such a network, the execution of a leader election algorithm would result in the identification of only one node. If that node is the only node monitoring all other nodes, then the system no longer remains distributed but becomes centralized. Such a process will result in a single cluster and would require tremendous amount of transmission and computation power. Hence, the modified version of the leader election scheme is used. Instead of choosing only one leader, multiple nodes can be elected to perform the duties of leader. Each of the leader would be a leader with respect to a subset of the nodes- that is, it would have dominance over a set of nodes. Each leader would know who belongs to the set. This gives rise to the concept of clustering.

Within each cluster, nodes can communicate with each other in at most d hops. The clusters can be constructed based on node ID. The following algorithm partitions the multi hop network into some non-overlapping clusters. The following operational assumptions underlying the construction of the algorithm in a radio network. These assumptions are common to most radio data link protocols.

1. Every node has a unique ID and knows the ID's of its own. This can be provided by a physical layer for mutual location and identification of radio nodes
2. A message sent by a node is received correctly within a nite time by all of its 1 hop neighbors
3. Network topology does not change during the algorithm execution

The objective of the proposed clustering algorithm is to and an interconnected set of clusters covering the entire node population, namely, the system topology is divided into small partitions (clusters) with independent control. A good clustering scheme will tend to preserve its structure when a few nodes are moving and the topology is slowly changing. Otherwise, high processing and communication overheads will have to be paid to reconstruct clusters. Within a cluster, it should be easy to schedule packet transmissions and to allocate the bandwidth to real-time traffic. Across clusters, the spatial reuse of codes must be exploited.

Throughout this chapter we write G(V; E), or simply G, to denote a graph on a finite non empty set V of nodes having its edge set E. All the graphs considered in this paper are simple finite undirected and connected. The terms not defined here may be found. For any two nodes u and v the distance between u and v, denoted by d(u; v), is the length of the shortest path between them. For a given graph G, there are a number of properties related to distance between two nodes which have been widely studied by various authors.

Algorithm to Check whether the Metric

Dimension of a Given Graph G is Two

The following algorithm follows from the Theorem.

Step 1: Input: distance matrix
Input is the distance matrix ordered according to vertices of a graph G which is not a path.
Step 2 : Check if (G) 6= 2 from number of elements in V (G)

If number of vertices in G i.e.,jV (G)j > $(D 1)^2$ + 8where D is the diameter of the graph G then (G) 6= 2 (Samir Khuller et al).

Step 3: Selection of vertices for nding metric basis
Select only those vertices in G with degree less than or equal to three (Samir Khuller et al).

Step 4: Formation of distance partitions

Form distance partition fV_{i0}; V_{i1}; : : : V_{iki} g of V (G) with reference to every vertex v_i having degree less than or equal to three 1 i n.

Step 5: Identify the pair of vertices for finding metric basis

Given a pair (u_1; u_2) if eccentricity of a vertex u_2 is less than number of vertices at distance d(u_1; u_2)

and vice versa then fu_1; u_2g cannot be a metric basis for G. Consider only a remaining pairs.

Among the pairs (u_i; u_j) remaining, consider only the pairs with unique shortest path between them.

Step 6: Find intersection If there exists vertices v_i and v_j(i 6= j) with jV_{ik} \ V_{jl} j 1 for every k and l with 1 k e(v_i) and 1 l e(v_j), then fv_i; v_jg is a metric basis for the graph G. Otherwise the metric dimension of G is not equal to two.

Results

This section establishes some results pertaining to the structure of a graph G with (G) = 2. Let G be a graph with (G) = 2 and fv_1; v_2g be a metric basis of G. Further, let fV_0; V_1; V2; : : : ; V_K g be the distance partition of G with reference to the vertex v_1. The results of the Theorem are due to Samir Khuller et al and a simple alternative proof using the concept of distance partition is given.

Theorem: For any vertex v 2Vj there exists a shortest path of length j between v_1 and v. In fact, a shortest path from v1 to v contains exactly one vertex w_1 2 Vi for 1 i j, and the distance d(w_1; v) = j i.

Proof. The first part of the theorem is immediate from the definition of distance partite set and v $_2$ V_j. Note that if u_1; u_2 are adjacent and u_1 2 V_i for some i 1, then u_2 is in one of Vi 1; Vi and V_i+1. Suppose that a shortest path from v1 to v 2 V_j of length j consists of more than one vertices u_1; u_2 2 Vi where 1 i < j . Then the shortest path is of the form v1; w1; : : : ; u_1; : : : u_2; : : : ; v since d(v_1; v) = j, j = length(v_1 u_1) + length(u_1 u_2) + length(u_2) v > d(v_1; u_1)+length(u_2 v). Since u_1; u_2 2 V_i, we have d(v_1; u_1) = d(v_1; u_2) = i. So, there exists a path v1 to u_2 of length i. Hence we obtain a path (v_1 u_2)[(u_2 v) of length less than j from v_1 to u this contradict d(v_1; v) = j.

Theorem: If G is a graph with (G) = 2 and metric basis fv_1; v_2g, then there exists a unique shortest path between v_1 and v_2.

Proof. Let V_0; V_1; V_2; : : : ; V_K be the distance partite sets with reference to v_1 and v_2 2 V_j. By Theorem: the shortest path between v_1 and v_2 contains only one vertex from each distance partite set V_0; V_1; V_2; : : : ; V_j 1. Suppose that p_1 and p_2 are two distinct paths between v1 and v_2. Let V_i be the rst partite set, while moving from v_2 to v_1, in which p_1 and p_2 pass through two distinct

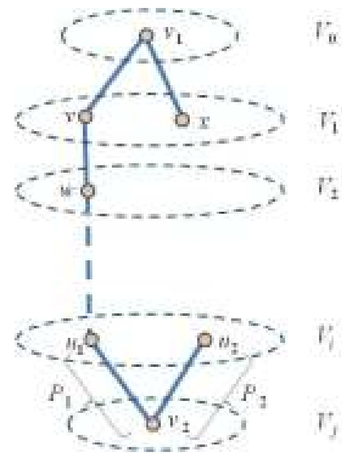

Figure : Unique shortest path between v_1 and v_2

vertices u_1 and u_2 respectively. Then d(v_2; u_1) = d(v_2; u_2) and hence u1 and u_2 are not resolved by any of v_1 and v_2, a contradiction to the fact that fv_1; v_2g is a metric basis of G.

Theorem Let fv_1; v_2g be a metric basis of G with (G) then degree of both v_1 and v_2 is less than or equal to three.

Proof. Let d(v_1; v_2) = d. Then any vertex adjacent to v1 is at distance d

1; dord + 1 from v_2. Since any pair of vertices that are adjacent to v_1 are not resolved by v1 and are to be resolved by v_2, the distances from these vertices to v_2 are different. Hence the number of vertices adjacent to v_1 does not exceed three. In other words, deg(v_1) 3. Similarly deg(v_2)3.

Theorem: Let fv_1; v_2g be a metric basis of G where (G) = 2. For any vertex u on the unique shortest path between v_1 and v_2, there exists at most one vertex adjacent to it in the distance partite set with respect to v_1 to which it belongs to. Further, u has exactly one vertex adjacent to it in the preceding distance partite set.

Conclusion

In this chapter, the distance partition of vertex set of a graph G with reference to a vertex in it is denned and with the help of the same, a graph with metric dimension two (i.e.(G)= 2) is characterized. In the process, a polynomial time algorithm is developed which verifies that if the metric dimension of a given graph G is two. The same algorithm explores all metric bases of graph G whenever (G) = 2. The bound for cardinality of any distance partite set with reference to a given vertex, when ever (G) = 2 is found. Also, in a graph G with (G) = 2, a bound for cardinality of any distance partite set as well as a bound for number of vertices in any sub graph H of G is obtained in terms of diam H.

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